M. Math IInd Yr Final Exam 2016 Advanced Functional Analysis

Answer all the questions. Each question is worth 6 points. You may state correctly and use any result proved in the class. However if an answer is an almost immediate consequence of the stated result, such a result also needs to be proved.

All topological spaces are assumed to be Hausdorff.

1) Let X be a complex normed linear space. Suppose $\mathcal{L}(X)$, the space of bounded operators, equipped with the usual norm, is a Banach space. Show that X is a Banach space.

2) Let X be a normed linear space. Let $\{x_n\} \subset X$ ~~is~~ be a sequence such that, the closed convex hull $K = \overline{CO(\{x_n\})}$ is a compact set. Let $x_0 \in X$ and let $\mu = \sum \frac{1}{2^n} \delta(x_n)$. Suppose $x^*(x_0) = \int_K x^* d\mu$ for all $x^* \in X^*$. Show that $x_0 \in K$.

3) Let X be a locally convex topological vector space. Let $F \subset X^*$ be a finite dimensional subspace. Show that there is a closed subspace $Y \subset X$ such that $Y^\perp = F$.

4) Let X, Y be LCTVS spaces. Let $T : X \to Y$ be an isomorphism. Suppose X^* and Y^* are equipped with the weak*-topology. Show that $T^* : Y^* \to X^*$ is an isomorphism.

5) Give examples of two normed linear spaces, and a continuous linear map T between them such that T has closed range but the range of T^* is not closed.

6) Show that the space of regular Borel probability measures on $[0, 1]$, equipped with the weak*-topology is a metrizable space.

7) Let $D = \{z : |z| < 1\}$ be the open unit disc. Let $A(D)$ denote the space of bounded analytic functions on D equipped with the supremum norm. ~~Let~~ Fix $n \geq 1$, let $F = \{p \in A(D) : p \textit{ is a polynomial of degree atmost } n\}$. Show that $A(D) = Y \bigoplus F$ (direct sum) for some closed subspace $Y \subset A(D)$. Let $P : A(D) \to Y$ be the canonical projection. Show that P is not a compact operator.

8) Let X be a normed linear space. State and prove the Banach-Alaoglu theorem.

9) Let $T : \ell^2 \to \ell^2$ be defined by $T(\{\alpha_n\}) = \{\frac{\alpha_n}{n}\}$. Show that T is a compact operator.

10) Let $T \in \mathcal{L}(X)$ be such that T^* maps extreme points of the unit ball of X^* to extreme points of the unit ball of X^*. Show that T is an extreme point of the unit ball of $\mathcal{L}(X)$.